Solution of HW7

1 The forward equation is

$$\begin{cases} P'_{x0}(t) = -\lambda P_{x0}(t) + \mu P_{x1}(t), \\ P'_{x1}(t) = \lambda P_{x0}(t) - \mu P_{x1}(t), \end{cases} \qquad x = 0, 1,\ t \geq 0.$$

The rate matrix $D = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$. The eigenvalue of D are 0 and $-(\lambda+\mu)$ with their corresponding eigenvectors $(1,1)^t$ and $(\lambda, -\mu)^t$ respectively.
Write $Q = \begin{pmatrix} 1 & \lambda \\ 1 & -\mu \end{pmatrix}$. Then

$$Q^{-1}DQ = \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix}.$$

Hence the transition functions are given by

$$\begin{aligned} P(t) = e^{Dt} &= Q \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{pmatrix} Q^{-1} \\ &= \begin{pmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \end{pmatrix}, \qquad t \geq 0. \end{aligned}$$

2 The rate matrix is given by

$$D = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 \\ 0 & \lambda_0 & -\lambda_0 \end{pmatrix}.$$

The forward equation is

$$\begin{cases} P'_{x0}(t) = -\lambda_0 P_{x0}(t) + \mu_1 P_{x1}(t), \\ P'_{x1}(t) = \lambda_0 P_{x0}(t) - (\lambda_1+\mu_1) P_{x1}(t) + \lambda_0 P_{x2}(t), \\ P'_{x2}(t) = \lambda_1 P_{x1}(t) - \lambda_0 P_{x2}(t), \end{cases} \qquad x = 0, 1, 2,\ t \geq 0.$$

Note also that $P_{x0}(t) + P_{x1}(t) + P_{x2}(t) \equiv 1$, putting it into the second equation, we have

$$P_{x1}(t) = \lambda_0 - (\lambda_0 + \lambda_1 + \mu_1) P_{x1}(t), \quad x = 0, 1, 2,\ t \geq 0.$$

With initial condition $P_{01}(0) = 0$, we solve that

$$P_{01}(t) = \frac{\lambda_0}{\lambda_0+\lambda_1+\mu_1} - \frac{\lambda_0}{\lambda_0+\lambda_1+\mu_1} e^{-(\lambda_0+\lambda_1+\mu_1)t}.$$

Put this solution into the first equation, we have

$$P'_{00}(t) = -\lambda_0 P_{00}(t) + \frac{\mu_1\lambda_0}{\lambda_0+\lambda_1+\mu_1} - \frac{\mu_1\lambda_0}{\lambda_0+\lambda_1+\mu_1} e^{-(\lambda_0+\lambda_1+\mu_1)t}.$$

With the initial condition $P_{00}(0) = 1$,

$$P_{00}(t) = \frac{\mu_1}{\lambda_0+\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-\lambda_0 t} + \frac{\lambda_0\mu_1}{(\lambda_0+\lambda_1+\mu_1)(\lambda_1+\mu_1)}e^{-(\lambda_0+\lambda_1+\mu_1)t}.$$

Finally,

$$\begin{aligned} P_{02}(t) &= 1 - P_{00}(t) - P_{01}(t) \\ &= \frac{\lambda_1}{\lambda_0+\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1}e^{-\lambda_0 t} + \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1+\mu_1)(\lambda_1+\mu_1)}e^{-(\lambda_0+\lambda_1+\mu_1)t}. \end{aligned}$$

10 **(a)** The forward equation is

$$\begin{cases} P'_{xy}(t) = -\mu_y P_{xy}(t) + \mu_{y+1}P_{x,y+1}(t), & y \le x-1, \\ P'_{xx}(t) = -\mu_x P_{xx}(t), & y = x. \end{cases}$$

(b) Directly solve the second equation with initial condition $P_{xx}(0) = 1$,

$$P_{xx}(t) = e^{-\mu_x t}.$$

(c) For $x = y$, it is done in (b). For $x < y$, $P_{xy}(t) = 0$. Now only consider the case $x > y$. Multiplying the integrating factor $e^{\mu_y t}$ on both sides in the first equation, we obtain

$$(e^{\mu_y t}P_{xy}(t))' = \mu_{y+1}e^{\mu_y t}P_{x,y+1}(t).$$

Integrating both side, and note that $P_{xy}(t) = 0$ for all $x > y$, we have

$$P_{xy}(t) = \mu_{y+1}\int_0^t e^{-\mu_y(t-s)}P_{x,y+1}(s)ds.$$

(d) Put $y = x-1$ and the solution in (b) into the equation in (c), and then integrate directly,

$$P_{x,x-1}(t) = \begin{cases} \frac{\mu_x}{\mu_{x-1}-\mu_x}(e^{-\mu_x t} - e^{-\mu_{x-1}t}), & \mu_{x-1} \neq \mu_x, \\ \mu_x t e^{-\mu_x t}, & \mu_{x-1} = \mu_x. \end{cases}$$

(e) This is proved directly by backward mathmatical induction on y from x to 0. It clearly holds by (b) when $y = x$. Assume it hols for $y+1$, then for y, by (c),

$$\begin{aligned} P_{xy}(t) &= (y+1)\mu\int_0^t e^{-y\mu(t-s)}\binom{x}{y+1}(e^{-\mu s})^{y+1}(1-e^{-\mu s})^{x-y-1}ds \\ &= (y+1)\binom{x}{y+1}\mu e^{-y\mu t}\int_0^t e^{-\mu s}(1-e^{-\mu s})^{x-y-1}ds \\ &= (y+1)\binom{x}{y+1}(e^{-\mu t})^y\int_1^{e^{-\mu t}}(1-u)^{x-y-1}du \\ &= (y+1)\binom{x}{y+1}(e^{-\mu t})^y\frac{(1-e^{-\mu t})^{x-y}}{x-y} \\ &= \binom{x}{y}(e^{-\mu t})^y(1-e^{-\mu t})^{x-y}. \end{aligned}$$

This completes the induction step.

12 **(a)** The forward equation is

$$\begin{cases} P'_{x0}(t) = \mu P_{x1}(t), \\ P'_{xy}(t) = \lambda(y-1)P_{x,y-1}(t) - (\lambda+\mu)yP_{xy}(t) + \mu(y+1)P_{x,y+1}(t), \quad y \geq 1. \end{cases}$$

(b) By the forward equation,

$$\begin{aligned} m'_x(t) &= \sum_{y=0}^{\infty} yP'_{xy}(t) = \sum_{y=1}^{\infty} yP'_{xy}(t) \\ &= \sum_{y=1}^{\infty} \big(\lambda y(y-1)P_{x,y-1}(t) - (\lambda+\mu)y^2 P_{xy}(t) + \mu y(y+1)P_{x,y+1}(t)\big) \\ &= \sum_{y=0}^{\infty} \lambda(y+1)yP_{xy}(t) - \sum_{y=1}^{\infty}(\lambda+\mu)y^2 P_{xy}(t) + \sum_{y=2}^{\infty} \mu(y-1)yP_{xy}(t) \\ &= \sum_{y=0}^{\infty} \big(\lambda(y+1)y - (\lambda+\mu)y^2 + \mu(y-1)y)\big)P_{xy}(t) \\ &= \sum_{y=0}^{\infty}(\lambda-\mu)yP_{xy}(t) = (\lambda-\mu)m_x(t). \end{aligned}$$

(c) Solving the ODE in (b) under the initial condition

$$m_x(0) = \sum_{y=0}^{\infty} yP_{xy}(0) = \sum_{y=0}^{\infty} \delta_{xy} \cdot y = x,$$

we get

$$m_x(t) = m_x(0)e^{(\lambda-\mu)t} = xe^{(\lambda-\mu)t}.$$

13 **(a)** By the forward equation,

$$\begin{aligned} s'_x(t) &= \sum_{y=0}^{\infty} y^2 P'_{xy}(t) = \sum_{y=1}^{\infty} y^2 P'_{xy}(t) \\ &= \sum_{y=1}^{\infty} \big(\lambda y^2(y-1)P_{x,y-1}(t) - (\lambda+\mu)y^3 P_{xy}(t) + \mu y^2(y+1)P_{x,y+1}(t)\big) \\ &= \sum_{y=0}^{\infty} \lambda(y+1)^2 yP_{xy}(t) - \sum_{y=1}^{\infty}(\lambda+\mu)y^3 P_{xy}(t) + \sum_{y=2}^{\infty} \mu(y-1)^2 yP_{xy}(t) \\ &= \sum_{y=0}^{\infty} \big(\lambda(y+1)^2 y - (\lambda+\mu)y^3 + \mu(y-1)^2 y)\big)P_{xy}(t) \\ &= \sum_{y=0}^{\infty} \big(2(\lambda-\mu)y^2 + (\lambda+\mu)\big)P_{xy}(t) \\ &= 2(\lambda-\mu)s_x(t) + (\lambda+\mu)m_x(t) \\ &= 2(\lambda-\mu)s_x(t) + (\lambda+\mu)xe^{(\lambda-\mu)t}. \end{aligned}$$

(b) Solving the ODE in (a) under the initial condition

$$s_x(0) = \sum_{y=0}^{\infty} y^2 P_{xy}(0) = \sum_{y=0}^{\infty} \delta_{xy} \cdot y^2 = x^2,$$

we get

$$s_x(t) = \begin{cases} \left(x^2 + \dfrac{\lambda+\mu}{\lambda-\mu}x\right) e^{2(\lambda-\mu)t} - \dfrac{\lambda+\mu}{\lambda-\mu}xe^{(\lambda-\mu)t}, & \lambda \neq \mu, \\ x^2 + 2\lambda xt, & \lambda = \mu. \end{cases}$$

(c) Under the condition $X(0) = x$,

$$\text{Var } X(t) = s_x(t) - (m_x(t))^2 = \begin{cases} \dfrac{\lambda+\mu}{\lambda-\mu}x(e^{2(\lambda-\mu)t} - e^{(\lambda-\mu)t}), & \lambda \neq \mu, \\ 2\lambda xt, & \lambda = \mu. \end{cases}$$

15 **(a)** Following page 101, $X_2(t)$ has a binomial distribution with parameters x and $e^{-\mu t}$. For $k \geq 0$,

$$\begin{aligned} P(X_2(t) = k) &= \sum_{x=k}^{\infty} \pi_0(x) P_x(X_2(t) = k) \\ &= \sum_{x=k}^{\infty} e^{-\nu} \frac{\nu^x}{x!} \binom{x}{k} (e^{-\mu t})^k (1 - e^{-\mu t})^{x-k} \\ &= e^{-\nu} \frac{(\nu e^{-\mu t})^k}{k!} \sum_{x=k}^{\infty} \frac{(\nu(1 - e^{-\mu t}))^{x-k}}{(x-k)!} \\ &= e^{-\nu e^{-\mu t}} \frac{(\nu e^{-\mu t})^k}{k!}. \end{aligned}$$

Hence $X_2(t)$ has a Poisson distribution with parameter $\nu e^{-\mu t}$.

(b) Since $X_1(t)$ and $X_2(t)$ are independent and have Poisson distribution with parameters $\frac{\lambda}{\mu}(1 - e^{-\mu t})$ and $\nu e^{-\mu t}$ respectively, the sum $X(t) = X_1(t) + X_2(t)$ has a Poisson distribution with parameter

$$\frac{\lambda}{\mu}(1 - e^{-\mu t}) + \nu e^{-\mu t} = \frac{\lambda}{\mu} + \left(\nu - \frac{\lambda}{\mu}\right) e^{-\mu t}.$$

(c) From (b), we know $X(t)$ has the same distribution as $X(0)$ if and only if

$$\nu = \frac{\lambda}{\mu} + \left(\nu - \frac{\lambda}{\mu}\right) e^{-\mu t} \quad \Longleftrightarrow \quad \nu = \frac{\lambda}{\mu}.$$